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On the removability of Berry's phase

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Abstract. In this paper we present a thorough analysis of the removability of the Berry phase. We show that for any system we can, by properly restricting the parameter space, turn the geometrical phase into an extra shift of the dynamical one. However, this extra shift in the dynamical phase retains its truly geometrical character. The removability is thus only apparent and the geometrical phase can still be measured in interference experiments, as indeed has already been done several times.

1. Introduction

When a Hamiltonian H depends on slowly changing parameters, the quantum dynamics is approximately described by the instantaneous eigenstates of H multiplied by a time-dependent phase. If H evolves in a cycle, so that $H(T) = H(0)$, this phase contains two parts: a dynamical phase, the integral of the instantaneous frequency $E_n(t)/\hbar$, and an additional phase (Berry 1984) which can be related to the geometry of a Hilbert space bundle over the base space of parameters (Simon 1983, Kiritsis 1987). The Berry phase depends only on the path taken in parameter space and not on the rate of progress along that path. It is gauge invariant, in the sense that we can freely redefine the phases of the instantaneous eigenstates without affecting it. However, there is another freedom due to the fact that different Hamiltonians can describe the same physical system. These Hamiltonians are related (classically) by canonical or (quantally) by unitary transformations, depending generally on time. It might seem that, using these time-dependent unitary transformations, Berry's phase can be removed; but that is not the case. The geometrical phase can be shifted into the dynamical part but it retains its geometrical character and indeed it can be measured experimentally.

The paper is organised as follows: in § 2 we recall the various techniques known from the literature for deriving the Berry phase and set up the formalism necessary to deal with the removability issue; we also call attention to the global aspects of the problem. In § 3 we further expand and clarify the theoretical framework developed in § 2 by means of some examples and experiments (actually performed or simply proposed). The appendix contains the analysis of the removability problem at the classical level.

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2. General framework

The starting point for our application of the adiabatic technique relies on the possibility of instantaneously diagonalising a time-dependent Hamiltonian $H(t)$ whose time evolution is slow enough so that no level transition is induced by the dynamics. To develop Berry's phases, we have to impose on our system the additional requirement that, after a period T, the Hamiltonian $H(t)$ comes back to its starting value $H(0)$:

$$H(T) = H(0). \tag{2.1}$$

$H(t)$ is instantaneously diagonalised as

$$H(t) = U(t)H_D(t)U^\dagger(t) \tag{2.2}$$

where $U(t)$ is a unitary operator and $H_D(t)$ is diagonal in a (fixed) orthonormal basis $\{|n\rangle\}$: $H_D(t) = \mathrm{diag}(E_1(t), E_2(t), \ldots)$. With no loss of generality we can impose the boundary condition $U(0) = I$. It is obvious from (2.2) that the instantaneous eigenstates of $H(t)$ are given by $U(t)|n\rangle$. Because of the cyclic property of $H(t)$, an eigenstate $U(T)|n\rangle$ of $H(T)$ can differ from $|n\rangle$ at most by a phase factor

$$U(T)|n\rangle = \exp(\mathrm{i}\alpha_n(T))|n\rangle. \tag{2.3}$$

From this last relation, as $U(T)$ and $H_D(0)$ are diagonal in the same basis $|n\rangle$, we deduce that

$$\left[U(T), H_D(0)\right] = 0. \tag{2.4}$$

Next we notice that the $U(t)$ are not uniquely determined by (2.2) and (2.4). Indeed we are still free to redefine $U(t)$ as

$$U(t) \to U(t)\Omega(t) \tag{2.5}$$

where $\Omega(t)$ is any unitary operator satisfying

$$\Omega(0) = I \qquad \left[\Omega(t), H_D(t)\right] = 0.$$

A possible way to fix $U(t)$ unambiguously is to impose on $U(t)$ the 'parallel transport' condition (Messiah 1981, Simon 1983, Anandan and Stodolsky 1987, Jordan 1988)

$$\langle n| U^\dagger(t)\dot{U}(t)|n\rangle = 0. \tag{2.6}$$

With the help of this further assumption, starting from a state $|n\rangle$ at $t = 0$ and exploiting the adiabatic hypothesis of no level crossing, we can solve the time-dependent Schrödinger equation and obtain (Messiah 1981)

$$|n, t\rangle \simeq \exp\left(-\frac{\mathrm{i}}{\hbar}\int_0^t E_n(\tau)\,\mathrm{d}\tau\right) U(t)|n\rangle.$$

Because of the cyclic property of $H(t)$, at time T we end up with the same initial state $|n\rangle$ multiplied by an overall phase $\mathrm{i}\Phi_n$:

$$|n, T\rangle = \exp(\mathrm{i}\Phi_n(T))|n\rangle$$

with

$$\Phi_n(T) = -\frac{1}{\hbar}\int_0^T E_n(t)\,\mathrm{d}t + \gamma_n(T) \tag{2.7}$$

where $\gamma_n(T)$ is given by $U(T)\,|n\rangle = \exp(\mathrm{i}\gamma_n(T))\,|n\rangle$. Note that $\gamma_n(T)$ is not the same as $\alpha_n(T)$ from (2.3), but is obtained from the parallel-transported $U(t)$ (2.6). We see from (2.7) that, besides the expected dynamical phase, we have a contribution of purely geometric origin, $\gamma_n(T)$, which is Berry's phase. We want to point out that it is *only* by virtue of the parallel transport constraint (2.6) that we can read the geometrical part of the total phase directly from the eigenvalues of $U(T)$†.

Let us now address the issue of unitary time-dependent transformations $W(t)$ acting on the vectors of our Hilbert space. The Schrödinger equation for the transformed states $|\tilde{\psi}(t)\rangle = W(t)\,|\psi(t)\rangle$ is

$$\mathrm{i}\hbar\frac{\partial}{\partial t}\,|\tilde{\psi}(t)\rangle = \tilde{H}(t)\,|\tilde{\psi}(t)\rangle$$

where

$$\tilde{H}(t) = H'(t) - \mathrm{i}\hbar W(t)\dot{W}^{\dagger}(t) \tag{2.8}$$

with

$$H'(t) = W(t)H(t)W^{\dagger}(t).$$

The adiabatic assumption requires that $W(t)$ not induce transitions among the eigenstates of $H'(t)$; then only the diagonal part of (2.8) should be considered, so that

$$\begin{aligned}\tilde{H}(t) &\simeq H'(t) - \mathrm{i}\hbar\sum_n P'_n(t)W(t)\dot{W}^{\dagger}(t)P'_n(t)\\ &\simeq U'(t)H'_D(t)U'^{\dagger}(t)\end{aligned}$$

where $P'_n(t)$ is the projector onto the nth state of $H'(t)$ and $U'(t) = W(t)U(t)$. We must stress at this point that now *only* $U'(t)$ must be subject to the parallel transport condition (2.6) and, for this reason, we are led to relax the same requirement for $U(t)$.

Starting from $\tilde{H}(t)$ we then obtain the analogue of (2.7) with the new phase $\Phi'_n(T)$ given by

$$\Phi'_n(T) = -\frac{1}{\hbar}\int_0^T (E_n(t) + E_n^{(w)}(t))\,\mathrm{d}t + \gamma'_n(T) \tag{2.9}$$

where $E_n^{(w)}(t) = -\mathrm{i}\hbar\,\langle n|\,U^{\dagger}(t)\dot{W}^{\dagger}(t)W(t)U(t)\,|n\rangle$ and $\gamma'_n(T)$ is again given by

$$U'(T)\,|n\rangle = \exp(\mathrm{i}\gamma'_n(T))\,|n\rangle \tag{2.10}$$

† Notice that if we do not restrict the $U(t)$ to satisfy (2.6) we obtain, of course, the same total phase (which is a quantity physically sensible and measurable) but the geometrical part of the phase is in the form $\gamma_n(T) - (1/\hbar)\int_0^T \mathrm{d}t\,\langle n|\,U^{\dagger}(t)\dot{U}(t)\,|n\rangle$, where $\gamma_n(T)$ has the same meaning but a different value from the one given in (2.7) (this kind of approach was used for example by Bouchiat (1987)). In this context it is easy to convince oneself that the substitution (2.5) leaves unchanged the geometrical phase.

(now $\tilde{E}_n(t) = E_n(t) + E_n^{(w)}(t)$ are the instantaneous eigenvalues of $\tilde{H}(t)$). Equation (2.7) explicitly shows that a unitary transformation changes the balance between dynamical and geometrical parts of the phase. Having established this general formalism it is easy to find that particular unitary transformation $W(t)$ which makes $\phi_n'(T)$ entirely dynamical; it is

$$W(t) = U^\dagger(t).$$

In this case $U'(t) = I$ so (2.4) and (2.6) are trivially satisfied and we can see from (2.10) that

$$\gamma_n'(T) = 0 \qquad \mod 2\pi.$$

With this transformation we have 'apparently' removed the geometrical part of the phase. However it is straightforward to show that $\Phi_n = \Phi_n'$ so what we have done is just a complete shift of the geometrical part into the dynamical one *but preserving the total phase.*

In his original paper Berry (1984) considered Hamiltonians depending on time only through a set of slowly varying parameters $R_l(t)$: $H(t) = H(\boldsymbol{R}(t))$. The assumption that H depends on time through a set of parameters R_l forces us to enlarge our 'local' notion of time-dependent Hamiltonians $H(t)$. If we look at our Hamiltonian $H(t)$ as depending simply on the evolution parameter t (time) it follows that in the adiabatic approximation the dynamics generates a local one-parameter group of transformations $(U(t))$. Writing $H(\boldsymbol{R})$ instead of $H(t)$ takes us from a local to a global analysis of the problem; in fact $H(\boldsymbol{R})$ in principle allows us to 'probe' not only a fixed path $\boldsymbol{R}(t)$ in parameter space (which may be relevant in a particular experiment) but the whole of parameter space. Probing the parameter space as a whole we might encounter the topological subtleties connected with its geometry. Namely, if we try to instantaneously diagonalise $H(\boldsymbol{R})$,

$$H(\boldsymbol{R}) = U(\boldsymbol{R})H_D(\boldsymbol{R})U^\dagger(\boldsymbol{R})$$

we might be unable to define $U(\boldsymbol{R})$ globally over the parameter space (Kiritsis 1987, Gozzi and Thacker 1987b). Supposing for the moment that there are no topological obstructions to defining $U(\boldsymbol{R})$ globally, let us rephrase the issue of removability in terms of an H depending on R_l. In this picture the unitary operator $U(\boldsymbol{R})$ ceases in general to fulfil the parallel transport condition (2.6), while the cyclic condition (2.1) can be obtained by moving on closed loops in R_l so that (2.4) is automatically satisfied since $U(\boldsymbol{R}(T)) = U(\boldsymbol{R}(0)) = I$. We still have the freedom of making the substitution indicated in (2.5) with $\Omega(t) = \Omega(\boldsymbol{R}(t))$; this now amounts to multiplying the instantaneous eigenvalues of $H(\boldsymbol{R})$ by a parameter-dependent phase factor and corresponds exactly to a gauge transformation which does not affect the geometrical part of the phase (Berry 1984). The total phase in this description takes the form:

$$\Phi_n(T) = -\frac{1}{\hbar}\int_0^T E_n(\boldsymbol{R})\,\mathrm{d}t + \mathrm{i}\oint \mathrm{d}\boldsymbol{R}\,\langle n|\,U^\dagger(\boldsymbol{R})\nabla_{\boldsymbol{R}}U(\boldsymbol{R})\,|n\rangle\,. \tag{2.11}$$

We recognise in the second term on the right-hand side the geometrical part of the phase written as a contour integral of a 1-form (Berry's connection) defined over parameter space.

In order to remove the phase we perform on the states the usual unitary transformation effected by $W(t)$, now depending on time only through the cyclic R_l: $W(t) = W(\boldsymbol{R}(t))$. The new phase is then

$$\Phi'_n(T) = -\frac{1}{\hbar}\int_0^T (E_n(\boldsymbol{R}) + E_n^{(w)}(\boldsymbol{R}))\mathrm{d}t + \mathrm{i}\oint \mathrm{d}\boldsymbol{R}\,\langle n|\,(W(\boldsymbol{R})U(\boldsymbol{R}))^{\dagger}\nabla_{\boldsymbol{R}}(W(\boldsymbol{R})U(\boldsymbol{R}))\,|n\rangle$$

where $E_n^{(w)}$ has the same form as in (2.9). Once again the choice $W(\boldsymbol{R}) = U^{\dagger}(\boldsymbol{R})$ would make the Berry phase entirely dynamical. So also in this picture, if there are no topological obstructions, we can 'apparently' remove the Berry phase. However, we stress that the extra dynamical part $-(1/\hbar)\int_0^T E_n^{(w)}(\boldsymbol{R})\mathrm{d}t$ keeps its geometrical character; indeed we see from the expression for $E_n^{(w)}$, that the integral of $E_n^{(w)}(\boldsymbol{R})$ over time does not depend on the rate of progress around the loop since $E_n^{(w)}$ contains only the first derivative with respect to t of $\boldsymbol{R}$. In this sense it depends only on the 'geometry' of the loop while the same does not hold for $E_n(\boldsymbol{R})$.

Returning to the issue of removing the Berry phase *globally*, we note once again that it concerns the possibility of defining the operator $U(\boldsymbol{R})$ (which instantaneously diagonalises $H(\boldsymbol{R})$) globally over the parameter space. If the Hilbert space bundle defined by the Berry connection is non-trivial we cannot do this. But even if the bundle is trivial, the geometrical phase does not disappear, it is simply shifted into the dynamical one and moreover it retains its geometrical character†.

In the next section we illustrate the previous points with reference to hypothetical or actual experiments.

3. Examples and experiments

As a first example, consider the parametric harmonic oscillator (Hannay 1985, Berry 1985). The Hamiltonian is

$$H = \tfrac{1}{2}[\alpha p^2 + \beta(pq + qp) + \gamma q^2]$$

where α, β, and γ are slowly varying parameters, $\alpha\gamma > \beta^2$ and $\alpha > 0$. The energy levels are $E_n = (n + \frac{1}{2})(\alpha\gamma - \beta^2)^{1/2}$. The Berry phase for a closed circuit $\mathscr{C}$ in α, β and γ is

$$\gamma_n(\mathscr{C}) = (n + \tfrac{1}{2})\oint_{\mathscr{C}} \frac{\beta\mathrm{d}\alpha - \alpha\mathrm{d}\beta}{2\alpha(\alpha\gamma - \beta^2)^{1/2}}. \tag{3.1}$$

(This extra phase could perhaps be detected in an interference experiment similar to the one described by Rohrlich (1988) but with a charged particle.) Jackiw (1988) and de Sousa Gerbert (1989) add the total time derivative of $-\beta q^2/2\alpha$ to the classical Lagrangian; this induces a canonical transformation to a new Hamiltonian $\tilde{H}$ with no

† Strictly speaking, all these considerations apply only if an 'effective' parameter space is considered. These effective parameters are singled out by means of an algebraical approach that consists of closing the algebra to which $H(\boldsymbol{R})$ belongs. In this way $H(\boldsymbol{R})$ is, in general, embedded in a wider class of Hamiltonians showing explicitly their dependence on the effective parameters. For more details on this subject see Giavarini *et al* (1989).

cross term in p and q and γ replaced by $\gamma - \beta^2/\alpha - (\mathrm{d}/\mathrm{d}t)(\beta/\alpha)$. At the quantum level, we obtain $\tilde{H}$ from H via the unitary transformation $W = \exp(\mathrm{i}\beta q^2/2\alpha)$. By applying (3.1) we can verify that no Berry phase arises from $\tilde{H}$. The phase is present, however, in a new guise: the instantaneous energy levels are now

$$\tilde{E}_n \simeq \left(n + \tfrac{1}{2}\right)(\alpha\gamma - \beta^2)^{1/2}\left[1 - \frac{\alpha}{2(\alpha\gamma - \beta^2)}\frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{\beta}{\alpha}\right) + \ldots\right]$$

where the dots refer to terms with two or more time derivatives of the slow parameters. The difference between $\tilde{E}_n$ and E_n, when integrated over time as part of the dynamical phase, is equivalent to the Berry phase in (3.1). de Sousa Gerbert (1989) notes this shift and suggests that the Berry phase loses its geometrical significance when it can be cancelled globally. The Berry phase has been removed in the sense that the quantum adiabatic theorem, without Berry's modification, correctly accounts for the extra phase. But even though the extra phase is now dynamical, it retains its geometric character: it depends only on the (closed) path in parameter space and not on the rate of progress along the path. The adiabatic assumption is essential here; note by comparison that if we make the generalised phase of Aharonov and Anandan (1987) dynamical, terms with more than one time derivative cannot be neglected. Then it is more problematic to decide what is geometrical.

A simple example of global removability of the Berry phase is given by the 'displaced harmonic oscillator' (Chaturvedi *et al* 1987), a system described by the Hamiltonian

$$H = (p - \beta)^2 + (q - \alpha)^2$$

where $\alpha, \beta \in \mathbb{R}$ are the adiabatic parameters. The instantaneous eigenvalues of H are the odd integers $E_n = 2n + 1$ and do not depend on the external parameters. The Berry phase is known to be the same for all the eigenstates

$$\gamma(\mathscr{C}) = -\tfrac{1}{2}\oint_{\mathscr{C}}(\alpha\,\mathrm{d}\beta - \beta\,\mathrm{d}\alpha).$$

The parameter space ($\mathbb{R} \times \mathbb{R}$) is flat and topologically trivial. Thus it is straightforward to find a unitary operator $U(\alpha, \beta)$, globally defined over the parameter space, that diagonalises instantaneously the Hamiltonian. We rewrite H as

$$H = U(\alpha, \beta)(p^2 + q^2)U^{\dagger}(\alpha, \beta)$$

where

$$U(\alpha, \beta) = \exp(-\mathrm{i}\alpha p + \mathrm{i}\beta q).$$

At this point it is natural to remove the geometrical phase by means of the unitary transformation $W(\alpha, \beta) = U^{\dagger}(\alpha, \beta)$, leading to the new Hamiltonian

$$\tilde{H} \simeq p^2 + q^2 + \tfrac{1}{2}(\alpha\dot{\beta} - \dot{\alpha}\beta),$$

where non-diagonal contributions have been dropped. This transformation amounts to renormalising the eigenvalues E_n by the quantity $\frac{1}{2}(\alpha\dot{\beta} - \dot{\alpha}\beta)$; the extra contribution to the energy levels of the oscillator, when integrated over time, yields Berry's phase. (At

the classical level the canonical transformation corresponding to $W(\alpha, \beta)$ is generated by the function $F(P, q; \alpha, \beta) = (q - \alpha)(\beta - P) + \frac{1}{2}\alpha\beta$.)

Among the papers reporting measurements of Berry's phase are several that note the freedom to change the form of the Hamiltonian. The Berry phase of a spinning particle in a magnetic field (Berry 1984) was measured by Bitter and Dubbers (1987) among others. They used beams of polarised neutrons in a twisting magnetic field. The twisting field represented an external parameter going through its adiabatic cycle, and the Berry phase is proportional to the solid angle swept out. This experiment does not exactly fit Berry's definition since the neutrons were not polarised along the direction of the field. The geometrical principle is nevertheless the same and the predicted phase shift was observed as an additional rotation of the precessing neutrons. The authors remark that '*in its simplest manifestation, which we believe to have realised, the appearance of a topological phase seems to be trivial: it can be generated or transformed away by going to a rotating-reference frame...*'. It should be noted that their experimental setup did not permit the magnetic field to point directly along the beam axis. By excluding a region from the parameter space, they let their experiment probe only *local trivialisations* of the $U(1)$ phase bundle, which in this case is a monopole bundle. This topological issue can be kept separate, however, because even if the bundle is trivial, the geometric phase can still be detected.

For this specific experiment we have to consider the quantum Hamiltonian

$$H = \tfrac{1}{2}\hbar\boldsymbol{\sigma}\cdot\boldsymbol{B}$$

but with the parameter space (which in this case is a sphere) restricted by removing the south pole. Then we can rewrite H in the following way:

$$H = \tfrac{1}{2}\hbar B U(\zeta)\sigma_3 U^{\dagger}(\zeta) \tag{3.2}$$

where

$$B = |\boldsymbol{B}| \qquad U(\zeta) = (1 + |\zeta|^2)^{-1/2}\begin{pmatrix} 1 & \zeta \\ -\bar{\zeta} & 1 \end{pmatrix} \tag{3.3}$$

and $\zeta = -\exp(-\mathrm{i}\varphi)\tan\theta/2$ corresponds to the stereographic projection of the sphere from its south pole to the complex plane ζ (θ and φ indicate the direction of the magnetic field B). We observe that ζ is not well defined for $\theta = \pi$, that is the south pole; this is just the above-mentioned 'restriction' of the parameter space: the systems described by (3.2) live in a local trivialisation of the monopole bundle. Now, following the general prescription of § 2, the transformation needed to remove the phase is given by $W(\zeta) = U^{\dagger}(\zeta)$ so that the new Hamiltonian is

$$\tilde{H} \simeq \tfrac{1}{2}\hbar B\sigma_3 - \mathrm{i}\hbar U^{\dagger}(\zeta)\frac{\mathrm{d}}{\mathrm{d}t}U(\zeta)\Big|_D$$

where the subscript D indicates that, in accordance with the adiabatic approximation, only the diagonal part should be taken into account. It is simple algebra to verify that

$$U^{\dagger}(\zeta)\,\mathrm{d}U(\zeta)\Big|_D = \frac{\zeta\,\mathrm{d}\bar{\zeta} - \bar{\zeta}\,\mathrm{d}\zeta}{1 + |\zeta|^2}\,\frac{1}{2}\sigma_3 = \mathrm{i}(1 - \cos\theta)\mathrm{d}\varphi\,\frac{1}{2}\sigma_3. \tag{3.4}$$

The final form for the Hamiltonian is then

$$\tilde{H} \simeq \tfrac{1}{2}\hbar[B + (1 - \cos\theta)\dot{\phi}]\sigma_3. \tag{3.5}$$

In this new frame the magnetic field has the apparent magnitude $B' = B + (1 - \cos\theta)\dot{\phi}$ and points along σ_3, so it cannot sweep a solid angle. Consequently there is no additional geometrical contribution to the phase. However the precession rate of the neutrons is proportional to the new field B' and the extra term $(1 - \cos\theta)\dot{\phi}$ reproduces the Berry connection; the anholonomy is still present and is accounted for by an extra precession of the polarisation vector. We still call this extra term geometrical because it does not depend on the rate of progress along the loop. We can equally well perform the same kind of analysis at the classical level (see the appendix for details).

To convince the reader that this extra dynamical piece can actually be measured, we will analyse a couple of conceptual experiments based on interferences effects. These effects are crucial in singling out the geometrical part of the phase. In fact interference effects can be made insensitive to the frame (rotating or not) in which we measure them.

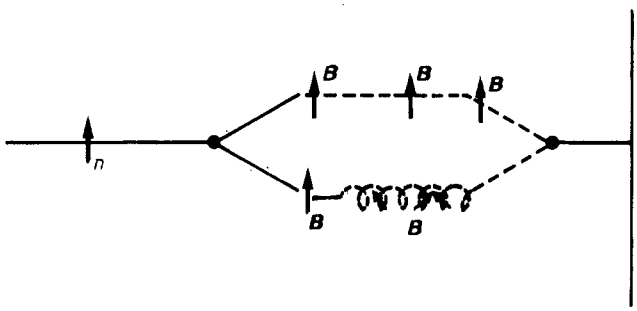

Figure 1. Berry's proposed experiment of a beam of neutrons that is split into two, one moving under a constant magnetic field B, and the other moving in a rotating B that describes a closed loop $\mathscr{C}$.

The first experiment is the one proposed by Berry (1984,1986). Take one beam of polarised neutrons and split it into two (see figure 1). One beam passes through an area with constant magnetic field $\boldsymbol{B}$; the other beam goes through a region where the magnetic field $\boldsymbol{B}$ is constant in magnitude but slowly, in a period T, winds around a circuit $\mathscr{C}$ subtending a solid angle $\Omega(\mathscr{C})$. The two beams are then recombined and the intensity $\mathscr{I}$ of the recombined beams is measured. After the period T the wavefunctions for the neutrons in the two beams (assuming the spin-up states) are given respectively by

$$\psi_1 = \exp(-\tfrac{1}{2}\mathrm{i}\,|\boldsymbol{B}|\,T) \qquad \psi_2 = \exp(-\tfrac{1}{2}\mathrm{i}\,|\boldsymbol{B}|\,T - \tfrac{1}{2}\mathrm{i}\Omega(\mathscr{C})).$$

The intensity of the recombined beams is

$$\mathscr{I} = |\psi_1 + \psi_2|^2 = \left|\exp(-\tfrac{1}{2}\mathrm{i}\,|\boldsymbol{B}|\,T) + \exp(-\tfrac{1}{2}\mathrm{i}\,|\boldsymbol{B}|\,T - \tfrac{1}{2}\mathrm{i}\Omega(\mathscr{C}))\right|^2$$
$$= \left[2 + 2\cos\left(\frac{\Omega(\mathscr{C})}{2}\right)\right].$$

From $\mathscr{I}$ we can measure the geometrical part $\Omega(\mathscr{C})/2$ of the phase. Let us now imagine that we go into a rotating frame moving with the magnetic field $\boldsymbol{B}$ felt by the second beam (by 'rotating frame' we mean the frame related to the matrix $U(\zeta)$ of (3.3)); we have then that the geometric part of the phase of the second beam is shifted into the dynamical one. The 'rotated' wavefunction is

$$\tilde{\psi}_2 = \exp\left(-\frac{\mathrm{i}}{\hbar}\int_0^T \tilde{E}(t)\,\mathrm{d}t\right)$$

where $\tilde{E}(t)$ is the one given by (3.5). From the frame of the rotating $\boldsymbol{B}$, the other neutron now feels a magnetic field rotating and describing a loop $-\mathscr{C}$ that subtends a solid angle $-\Omega(\mathscr{C})$; its wavefunction is then

$$\tilde{\psi}_1 = \exp\left(-\frac{\mathrm{i}}{\hbar}\int_0^T \tilde{E}(t)\,\mathrm{d}t + \tfrac{1}{2}\mathrm{i}\Omega(\mathscr{C})\right).$$

The new intensity $\mathscr{I}'$ is

$$\mathscr{I}' = |\tilde{\psi}_1 + \tilde{\psi}_2|^2 = \left[2 + 2\cos\left(\frac{\Omega(\mathscr{C})}{2}\right)\right] = \mathscr{I}.$$

So in the rotating frame the pattern of interference is still the same. This indicates a way to really single out the purely geometrical part: the trick is to have interference between the two systems and not simply to look at an isolated one.

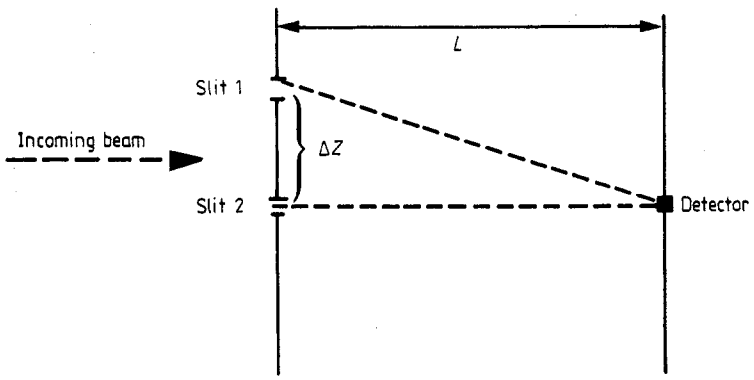

Figure 2. Double-slit experiment for neutrons moving in a B field that changes with z but is constant in x.

We can also envision the following more realistic experiment conceptually equivalent to the previous one. Imagine a double-slit experiment with neutrons in a magnetic field (see figure 2)

$$\boldsymbol{B}(z) = B[\hat{x}\sin\theta\cos(\varepsilon z) + \hat{y}\sin\theta\sin(\varepsilon z) + \hat{z}\cos\theta].$$

$\boldsymbol{B}$ rotates slowly, moving along the coordinate z, since ε is a small parameter (it plays the role of 'slowness' parameter). Let polarised neutrons, moving in the $\hat{x}$ direction, pass through slits lying along the z axis. In addition to the usual interference pattern

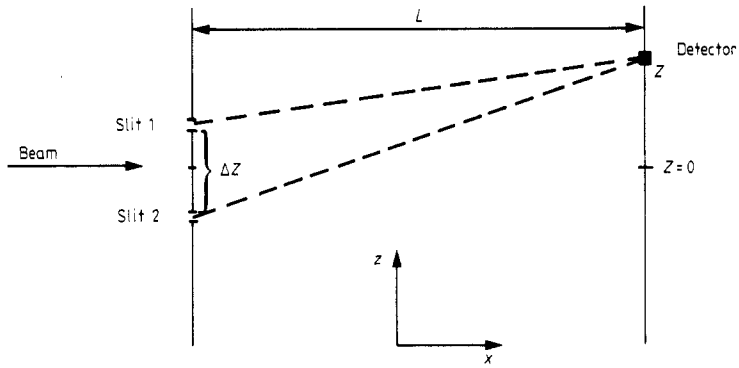

Figure 3. The same double-slit experiment as in figure 2 with the detector moved upward.

due to the difference in path lengths, there will be an extra shift because the neutrons have been rotated different amounts by the field, depending on the $\hat{z}$ component of their path. If we arrange the distance Δz between the slits so that $\varepsilon \Delta z = 2\pi$, then the different rotations of the beams correspond to one closed loop in parameter space. Berry's phase will appear as a shift in the pattern of fringes with respect to the case with constant $\boldsymbol{B}$. The Hamiltonian of the system (neglecting the $\hat{y}$ direction) is

$$H = \frac{1}{2m}(p_x^2 + p_z^2) + \tfrac{1}{2}\hbar\boldsymbol{\sigma}\cdot\boldsymbol{B}(z). \tag{3.6}$$

We can diagonalise H to order ε using the operator $U(\zeta)$ given in (3.3) with

$$\zeta = -\exp(-i\varepsilon z)\tan\theta'/2 \qquad \theta' = \theta + \frac{p_z}{mB}\varepsilon\sin\theta.$$

Thus

$$H \simeq U(\zeta)H_D U^{\dagger}(\zeta) + O(\varepsilon^2)$$

where H_D turns out to be

$$H_D = \frac{1}{2m}(p_x^2 + p_z^2) + \frac{\hbar B}{2}\sigma_3 + \frac{\hbar p_z \varepsilon}{2m}(1-\cos\theta)\sigma_3.$$

The first and the second terms are due, respectively, to the kinetic and magnetic energy of the neutrons; the third is the geometrical one: note in fact that it is $O(\varepsilon)$. From figure 3 it is easy to calculate the difference in path length at the point z and then to get the phase difference between the two beams. It will turn out to be the usual one of the two paths in a constant B field augmented† by the extra term present in H_D; so the overall pattern of fringes will be shifted by a factor of $-\pi(1-\cos\theta)$. Thus Berry's phase appears with the expected magnitude. If we want to observe the geometrical phase we have to compare the pattern of fringes in a constant $\boldsymbol{B}$ with the shifted one for the z-dependent $\boldsymbol{B}$. There is a difference between this example and the previous one, however: before, we changed to a reference frame that rotated with respect to both beams, and the effect on the phase in one beam was cancelled by the effect on the other. Here, the rotation is effected locally, because the unitary matrix $U(\zeta)$ depends on position and affects the two beams differently.

† In the experiment described in figure 1 (proposed by Berry) the two beams had the same path length so the only difference was of purely geometric origin.

These two conceptual experiments reflect the fact that indeed Berry's phase has been measured in real experiments. Examples include the NMR interferometry experiments (Suter *et al* 1988, Tycko 1987), the polarisation states experiments with lasers (Bhandari *et al* 1988, Simon *et al* 1988) or the experiment of Bitter and Dubbers (1987) itself. In particular Tycko (1987) discusses issues close to ours in showing how the geometric phase can appear in his experiment as 'fictitious' dynamical phase, but he can still measure it.

4. Conclusions

We have seen that Berry's phase can always be cancelled by a unitary transformation, once the parameter space has been suitably restricted (if necessary). However the phase reappears as dynamical and can in principle be measured unambiguously. Moreover it retains its geometrical character of depending only on the loop in parameter space and not on the rate of progress along it. Regarding the issue of global removability, the example of a spin in a magnetic field is one where the Berry phase cannot be removed globally, yet in all the experiments discussed above, the bundle became trivial by virtue of restrictions on the parameter space. A single measurement can, in general, probe only local trivialisations of these bundles and it is hard to imagine how any experiment could verify that a bundle is indeed non-trivial.

We conclude by observing that there is no conflict between this 'experimental' triviality and the fact that Berry's phase can still be detected and not taken away; experiments simply verify that the geometrical phase arises in a particular form, which corresponds to a section of a bundle (in general non-trivial). Moreover the form of the Berry connection is unchanged, though it shifts into the energy term. For example in the case of the spinning particle in a magnetic field, the Berry connection is $(1-\cos\theta)\mathrm{d}\varphi$ also when shifted into the dynamical phase, and it is just this form that cannot be extended over the full sphere of directions.

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Note added. After this work was completed, the authors became aware that possible ambiguities in defining the geometrical phase were pointed out some time ago (in a context different from ours) by Berry (1987) and more recently by Anandan (1989). In particular Anandan, as a solution to this problem, proposes an invariant formulation of the geometrical phase, describing the motion of the vectors in Hilbert space with respect to a chosen reference frame.

We can rephrase the discussion made at the end of § 2 in the language of Berry (1987). The total phase $\Phi(T)$, acquired by a state when $H(t)$ is taken slowly round a cycle, can be written as a Laurent series in the 'slowness' parameter ε (the inverse of the evolution period T):

$$\Phi(T) = \frac{1}{\varepsilon}\Phi_d + \gamma + \mathrm{O}(\varepsilon)$$

where transitions from the initial state to other states are neglected. For an adiabatic process ($\varepsilon \to 0$) only the first two terms on the right-hand side are significant: the first term ($\mathrm{O}(1/\varepsilon)$) is the dynamical contribution

to the phase, the second one (independent of ε) is Berry's phase (see (2.7)). The ε independence of the Berry phase is just another way to show that the geometrical phase depends only on the geometry of the path in parameter space and not on the rate of progress along it. Indeed, one way to unambiguously extract the geometrical contribution from the total phase Φ is via a limiting procedure:

$$\gamma \equiv \lim_{\varepsilon \to 0} \left(\Phi + \varepsilon \frac{\mathrm{d}\Phi}{\mathrm{d}\varepsilon} \right).$$

This definition applies equally well after we have performed a time-dependent unitary transformation (that is, to Φ' of (2.9)). Thus, even if the geometrical phase can apparently be absorbed into the dynamical one, this is another way to single it out unambiguously. The reason is that, after applying a time-dependent unitary transformation to remove Berry's phase, we end up with a new Hamiltonian $\tilde{H}(t)$ that is not merely a function of εt (as the original Hamiltonian was) but a function of both εt and ε alone. The coefficient of the $1/\varepsilon$ term in the phase now contains a part proportional to ε, producing a constant term to account for the original O(1) term (Berry's phase) that was removed. Thus $\tilde{H}(t)$ is in a sense a 'fake' Hamiltonian with geometry smuggled into its dynamics. In another sense, we can view this limiting procedure ($\varepsilon \to 0$) for defining the geometrical phase as a way to compare two Hamiltonians $\tilde{H}(t)$ 'nearby' in ε. Thus the Berry phase appears as a property not of an individual Hamiltonian but of the limit of a whole family of Hamiltonians indexed by the adiabatic parameter ε.

We thank one of the referees for suggesting this analysis to us.

Appendix

In this appendix we want to perform the same analysis presented in § 2 but for the classical case (Hannay 1985, Berry 1985). The anholonomy effect consists in this case of an extra angle (Hannay's angle) swept by the fast variables (p_i, q_i) while they are carried around in a closed loop by the slow parameters (Π_l, R_l). We will not review here the derivation of the Hannay angles but refer the reader to the literature (Hannay 1985, Berry 1985, Gozzi and Thacker 1987a, b, Giavarini *et al* 1989).

We have seen that in the quantum case we were free to carry out parameter-dependent unitary transformations; in the classical case we are free to make parameter-dependent canonical transformations. Following Gozzi and Thacker (1987b), the total geometrical plus dynamical part of the motion comes from the 'effective adiabatic action':

$$S_{\mathrm{eff}} = \int_0^T \mathrm{d}t \left[\left(\Pi_l + \left\langle p_i \frac{\partial q_i}{\partial R_l} \right\rangle \right) \dot{R}_l - \bar{H} \right]$$

where $i = 1, \ldots, N$; $l = 1, \ldots, M$ (repeated indexes are summed) and where

$$\bar{H}(I_i; R_l) = \mathscr{H}_1(I_i; R_l) + H_2(R_l, \Pi_l).$$

The p_i, q_i are the fast variables and Π_l, R_l the slow ones (or parameters). $\mathscr{H}_1(I; R)$ (we drop the subscripts from now on) is the Hamiltonian for the fast variables expressed in action-angle coordinates I and θ:

$$\mathscr{H}_1(I; R) = H_1(p(I, \theta; R), q(I, \theta; R); R).$$

$H_2(R_l, \Pi_l)$ is the Hamiltonian for slow variables. The 'average' symbol $\langle \cdots \rangle$ means the angle-average at fixed I and R:

$$\langle \mathscr{F}(q, p; R) \rangle = \frac{1}{(2\pi)^N} \int_0^{2\pi} \mathrm{d}\theta_1 \ldots \int_0^{2\pi} \mathrm{d}\theta_N \, \mathscr{F}(q(I, \theta; R), p(I, \theta; R); R). \tag{A1}$$

In the present work we are taking the slow variables to be external parameters, so we disregard the Lagrangian of the slow variables $L_2 = \Pi_l \dot{R}_l - H_2$ in S_{eff}. The classical analogue of the quantum phase Φ_n swept by the system in a period T is then given by

$$S_{\text{eff}} = \int_0^T \mathrm{d}t \left[\left\langle p \frac{\partial q}{\partial R} \right\rangle \dot{R} - \langle \mathscr{H}_1(I;R) \rangle \right]. \tag{A2}$$

In the equation above we have used the fact that $\mathscr{H}_1(I;R) = \langle \mathscr{H}_1(I;R) \rangle$ because $\mathscr{H}_1$ is independent of θ.

Let us now see what happens when we perform a time-dependent canonical transformation:

$$p \to \tilde{p} \qquad q \to \tilde{q}.$$

Suppose for example that we choose an F_1-type transformation (Goldstein 1950)†. Then H_1 goes into an $\tilde{H}_1$ given by

$$\tilde{H}_1(\tilde{p},\tilde{q};R) \equiv H_1\big(q(\tilde{q},\tilde{p};R),p(\tilde{q},\tilde{p};R);R\big) + \frac{\partial}{\partial t} F_1\big(q(\tilde{q},\tilde{p};R),\tilde{q};R\big).$$

Now in $\tilde{H}_1$ we re-express $\tilde{q}$, $\tilde{p}$ in terms of action-angle variables:

$$\tilde{q} = \tilde{q}(I,\theta;R) \qquad \tilde{p} = \tilde{p}(I,\theta;R)$$

and we have

$$\begin{aligned}\tilde{H}_1&\big(\tilde{q}(I,\theta;R),\tilde{p}(I,\theta;R);R\big) \\ &= H_1\big(q(I,\theta;R),p(I,\theta;R);R\big) + \frac{\partial}{\partial t} F_1\big(q(I,\theta;R),\tilde{q}(I,\theta;R);R\big).\end{aligned}$$

So the new $\widetilde{\mathscr{H}}_1(I,\theta;R)$ is

$$\widetilde{\mathscr{H}}_1(I,\theta;R) = \mathscr{H}_1 + \frac{\partial F_1}{\partial t}. \tag{A3}$$

Now, going back to equation (A2), we have to perform the change to $(\tilde{p},\tilde{q})$ inside $\langle p\mathrm{d}q \rangle$. From the general theory of canonical transformations (Goldstein 1950) we have that

$$p\,\mathrm{d}q = \tilde{p}\,\mathrm{d}\tilde{q} + \frac{\partial F_1}{\partial \tilde{q}}\,\mathrm{d}\tilde{q} + \frac{\partial F_1}{\partial q}\,\mathrm{d}q.$$

So, taking the average, we obtain

$$\langle p\,\mathrm{d}q \rangle = \langle \tilde{p}\,\mathrm{d}\tilde{q} \rangle + \left\langle \frac{\partial F_1}{\partial \tilde{q}}\,\mathrm{d}\tilde{q} + \frac{\partial F_1}{\partial q}\,\mathrm{d}q \right\rangle. \tag{A4}$$

If F_1 depends on t only through the parameters R_l ,we can write

$$\frac{\partial F_1}{\partial t} = \frac{\partial F_1}{\partial R}\dot{R}. \tag{A5}$$

† Nothing changes in the conclusions if it is not an F_1-type transformation.

Using now the relations (A3)–(A5) we can rewrite equation (A2) (for a closed loop in parameter space) as†

$$S_{\text{eff}} = \oint \left\langle p \frac{\partial q}{\partial R} \right\rangle \mathrm{d}R - \int_0^T \langle \mathscr{H}_1(I;R) \rangle \, \mathrm{d}t$$
$$= \oint \left\langle \tilde{p} \frac{\partial \tilde{q}}{\partial R} \right\rangle \mathrm{d}R - \int_0^T \left\langle \tilde{\mathscr{H}}_1(I,\theta;R) \right\rangle \mathrm{d}t$$
$$+ \int_0^T \left\langle \frac{\partial F_1}{\partial \tilde{q}} \frac{\mathrm{d}\tilde{q}}{\mathrm{d}R} + \frac{\partial F_1}{\partial q} \frac{\mathrm{d}q}{\mathrm{d}R} + \frac{\partial F_1}{\partial R} \right\rangle \dot{R} \, \mathrm{d}t.$$

The third term on the right-hand side of the equation above is

$$\Delta = \int_0^T \left\langle \frac{\partial F_1}{\partial \tilde{q}} \frac{\mathrm{d}\tilde{q}}{\mathrm{d}R} + \frac{\partial F_1}{\partial q} \frac{\mathrm{d}q}{\mathrm{d}R} + \frac{\partial F_1}{\partial R} \right\rangle \dot{R} \, \mathrm{d}t$$
$$= \int_0^T \left\langle \frac{\mathrm{d}F_1}{\mathrm{d}R} \right\rangle \dot{R} \, \mathrm{d}t = \oint \left\langle \frac{\mathrm{d}F_1}{\mathrm{d}R} \right\rangle \mathrm{d}R. \tag{A6}$$

This expression is zero on a closed loop in parameter space R, since‡ $\langle \mathrm{d}F_1/\mathrm{d}R \rangle = \mathrm{d}\langle F_1 \rangle / \mathrm{d}R$ and so

$$\Delta = \oint \frac{\mathrm{d}\langle F_1 \rangle}{\mathrm{d}R} \mathrm{d}R = 0$$

if $\langle F_1 \rangle$ is single-valued in R. It follows that S_{eff} (the sum of the geometrical $\int \langle p \, \mathrm{d}q \rangle$ and the dynamical part $-\int \mathscr{H}_1 \, \mathrm{d}t$) *is a canonical invariant* when evaluated on closed loops in parameter space. This is the analogue of the fact that the phase Φ_n in § 2 was a unitary invariant. Note that this is the *profound reason* why we need to add the extra geometrical part to the dynamical one: the dynamical part alone would not be canonically invariant; it is only the sum of the two that has this property. Note also, from (A4), that the geometrical part alone would not be invariant:

$$\oint \left\langle p \frac{\partial q}{\partial R} \right\rangle \mathrm{d}R \neq \oint \left\langle \tilde{p} \frac{\partial \tilde{q}}{\partial R} \right\rangle \mathrm{d}R$$

since $\langle p \, \partial q / \partial R \rangle$ does not change as a total derivative in R. The same can be said for the dynamical part (see (A3)). So, as in the quantum case, the non-invariance of the two single parts (geometrical and dynamical) allows us to shift as much as we want of the geometrical part into the dynamical one and vice versa, but the sum of the two remains the same. We could even make the geometrical part equal to zero (as in the quantum case) by performing a canonical transformation with an F_1 such that (see (A4))

$$\left\langle \frac{\partial F_1}{\partial q} \mathrm{d}q + \frac{\partial F_1}{\partial \tilde{q}} \mathrm{d}\tilde{q} \right\rangle = \langle p \mathrm{d}q \rangle . \tag{A7}$$

† The transformed $\tilde{\mathscr{H}}_1$ in principle might depend on θ, but thanks to the adiabatic hypothesis we will use its averaged $\langle \cdots \rangle$ value (Arnold 1978).

‡ Here we follow Berry (1985) and not Hannay (1985) or Gozzi and Thacker (1987b (appendix)). In Berry the $\langle \cdots \rangle$ averages are in the θ variables which do not depend on R (see the definition (A1)) and so $\mathrm{d}/\mathrm{d}R$ can be pulled out of the average $\langle \cdots \rangle$; while in Hannay (and also in the appendix of Gozzi and Thacker (1987b)) the averages are in the p, q variables, which depend on R, and so in that case the $\mathrm{d}/\mathrm{d}R$ cannot be pulled out of the average $\langle \cdots \rangle$.

Of course there might be obstructions in finding a globally defined $\langle F_1 \rangle$ that implements the relation (A7) (Gozzi and Thacker 1987b) but, as mentioned before, experiments probe local trivialisations of the associated bundle, so we can (also at the classical level) always take the geometrical part away locally, making it dynamical. The geometrical character of this extra piece of the dynamical phase is nevertheless preserved in the sense that, as in § 2, it just depends on the geometry of the loop and not on the rate of progress along it (see its expression in (A2)).

Let us notice from (A2) that, under a parameter-dependent canonical transformation, the term†

$$\mathscr{A}_l \, \mathrm{d}R_l \equiv \left(\left\langle p \frac{\mathrm{d}q}{\mathrm{d}R_l} \right\rangle - \langle \mathscr{H}_1 \rangle \left(\frac{\mathrm{d}t}{\mathrm{d}R_l} \right) \right) \mathrm{d}R_l$$

transforms as a connection in R_l space, in fact from (A3), (A5) and (A6), we have:

$$\mathscr{A}_l \to \mathscr{A}_l + \frac{\mathrm{d}\langle F_1 \rangle}{\mathrm{d}R_l}$$

(note that $\langle F_1 \rangle$ is, because of the average, a function of R and I only). So even the parameter-dependent canonical transformations can be seen as gauge transformations in R_l space. The connection is no longer just $\hat{\mathscr{A}}_l = \langle p \, \mathrm{d}q/\mathrm{d}R_l \rangle$ (as with the phase transformation of Berry (1985)), but

$$\mathscr{A}_l = \left\langle p \frac{\mathrm{d}q}{\mathrm{d}R_l} \right\rangle - \langle \mathscr{H}_1 \rangle \left(\frac{\mathrm{d}t}{\mathrm{d}R_l} \right).$$

The angle–origin reshifting (Berry 1985, Hannay 1985) is just a particular case of this parameter-dependent canonical transformation. The reader, however, might be bothered by the fact that in our new connection $\mathscr{A}_l$ we have a term $(\mathrm{d}t/\mathrm{d}R_l)$ that depends on the loop in parameter space chosen. To be precise our $\mathscr{A}_l$ is not a function of R_l but a functional of $R_l(t)$: once $R_l(t)$ is given then $\mathscr{A}_l$ becomes a function of R_l. So it is only in this loose sense that we can look at the parameter-dependent canonical transformations as 'gauge' transformations.

The rest of this appendix presents, at the classical level, the analysis given in § 2 for the $\boldsymbol{B} \cdot \boldsymbol{\sigma}$ model, made classical through Grassmannian variables.

Explicitly we have the classical Hamiltonian $H = -\frac{1}{2}\psi^{\dagger}\widehat{B}\psi$ where $\widehat{B}_{ij} = \mathrm{i}B_k \varepsilon_{kij}$ and $\psi = (\psi_1, \psi_2, \psi_3)$, $\psi_i^{\dagger} = \psi_i$, are real Grassmann variables (Casalbuoni 1976a, b, Berezin and Marinov 1977). Now suppose we perform a canonical transformation, generated by the function $F(\psi, \tilde{\psi})$, to new variables $\tilde{\psi}$

$$\tfrac{1}{2}\mathrm{i}\psi^{\dagger}\frac{\mathrm{d}}{\mathrm{d}t}\psi - H(\psi) = \tfrac{1}{2}\mathrm{i}\tilde{\psi}^{\dagger}\frac{\mathrm{d}}{\mathrm{d}t}\tilde{\psi} - \tilde{H}(\tilde{\psi}) + \frac{\mathrm{d}F}{\mathrm{d}t}.$$

Matching the coefficients we obtain the transformation rules:

$$\frac{\partial F}{\partial \psi_k} = \tfrac{1}{2}\mathrm{i}\psi_k^{\dagger} = \tfrac{1}{2}\mathrm{i}\psi_k \qquad \frac{\partial F}{\partial \tilde{\psi}_k} = -\tfrac{1}{2}\mathrm{i}\tilde{\psi}_k^{\dagger}$$

$$\tilde{H} = H + \frac{\partial F}{\partial t}.$$

† We will be inverting here the functional relations $R_l = R_l(t)$ to re-express t as a function of the R_l.

Before choosing the explicit form for F one observation is in order: in the adjoint representation of the su(2) algebra we have $(J_k)_{lm} = i\varepsilon_{klm}$ so that the magnetic-field matrix is given by $\hat{B} = \boldsymbol{B} \cdot \boldsymbol{J}$. Then we can describe the slow motion of B just as we have done in the quantum case (3.2), that is

$$\hat{B} = BU(\theta, \varphi) J_3 U^\dagger(\theta, \varphi) \qquad B = |\boldsymbol{B}|$$

where the U matrices are the analogue, in the adjoint representation (three dimensional), of the two-dimensional U matrices that we have already seen in their fundamental representation (3.3). Note that they are not globally defined over the parameter space. Now we fix the form of the generating function to be $F = \frac{1}{2} i \psi^\dagger U \tilde{\psi}$. This yields the Hamiltonian

$$\tilde{H} \simeq -\tfrac{1}{2} B \tilde{\psi}^\dagger J_3 \tilde{\psi} + \tfrac{1}{2} i \tilde{\psi}^\dagger U^\dagger \dot{U} \tilde{\psi} \Big|_D$$

where, according to the adiabatic hypothesis, we keep only the 'diagonal' part in the last term of the right-hand side. In order to calculate $U^\dagger \mathrm{d}U$ we use a group theoretical argument: the term in question belongs to the su(2) algebra so it must be of the form $U^\dagger \mathrm{d}U = \omega_k J_k$ where the coefficients ω_k are the Maurer–Cartan differential forms of su(2) which depend only on the abstract group algebra (Chevalley 1946). So we are entitled to evaluate them in the representation we like best; but we have already done this calculation in the fundamental representation when solving the quantum case (3.4): the diagonal term is $\omega_3 = i(1 - \cos\theta)\mathrm{d}\varphi$. The resulting Hamiltonian turns out to be

$$\tilde{H} \simeq -\tfrac{1}{2} [B + (1 - \cos\theta)\dot{\varphi}] \tilde{\psi}^\dagger J_3 \tilde{\psi}.$$

We see that $\tilde{H}$ is manifestly written in terms of 'normal modes' and the extra contribution to the magnetic field is responsible for the Hannay angle, the classical relic of the quantal Berry phase (Hannay 1985, Berry 1985).

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